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SOME PROPERTIES SIMILAR TO COUNTABLE COMPACTNESS AND LINDELÖF PROPERTY

Dušan Milovančević

Abstract. In this paper we further investigate the results given in [7], [8], [9]. In Section 2 we consider spaces X for which the closure of each countably compact (strongly countably compact, hypercountably compact) subspace of X has countably compact (strongly countably compact, hypercountably compact) property. In Section 3 we study some notions related to the classical concepts of being a Lindelöf, Menger or a Rothberger space.

1. Introduction

The closure of a subset A of a space X is denoted by $cl_X(A)$. In this paper we assume that all spaces are Hausdorff. For notions and definitions not given here see [6], [8], [12].

Definition 1.2. *A space X is said to be absolutely countably compact (ACC space) if for every open cover γ and every dense subspace $Y \subset X$ the cover $\gamma^{(1)} = \{St(x,\gamma) : x \in Y\}$ has a finite subcover of X (see [10]).*

It is clear that every ACC space is countably compact.

Definition 1.3. *Let X be a topological space.*

(a) *A space X is hypercountably compact (HCC) if every σ-compact set in X has compact closure in X (see [12]).*

(b) *A space X is strongly countably compact (SCC) if every countable subset in X has compact closure in X (see [8]).*

(c) *A space X is exponentially countably compact (ECC) if the hyperspace $exp(X)$ of X is a countably compact space (see [13]).*

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2. *CLCC*, *CLSC*, *CLHC*, *CLC* spaces

It can be shown that the closure of every pseudocompact subspace of a space X is a pseudocompact subspace of X. Since countable compactness implies pseudocompactness, the closure of every countably compact subspace of a space X is a pseudocompact subspace of X.

It is known that in the class of normal spaces countable compactness and pseudocompactness coincide, so that the closure of every countably compact subspace of a normal space X is a countably compact subspace of X.

In the class of ACC spaces every countably compact space is a closed subspace of some ACC space (see [10]).

In the class of first countable spaces strongly countable compactness and hypercountable compactness coincide, and every countably compact (SCC, HCC) subset of a first countable space X is closed in X. The proof of these facts is given in [12]. The following example shows that the first countability cannot be omitted from this assertion.

Let M be the Cartesian product of the interval $I = [0,1]$ with the usual topology and $[0,\omega_1]$, where $[0,\omega_1]$ is the space of ordinals less then or equal to the first uncountable ordinal with the order topology. Let $X = M \setminus \{(\alpha,\omega_1)\} : \frac{1}{3} \leq \alpha \leq \frac{2}{3}\}$ be a subspace of M. The space X is not countably compact since the subset $A = \left\{\left(\frac{2}{3}+\frac{1}{n},\omega_1\right) : n \in N\right\}$ of X has no limit point in X and X is not first countable. According to Proposition 2.8 in [12] the subspace $Y = [0,1] \times [0,\omega_1)$ of X is a countably compact (SCC, HCC) space and $cl_X(Y) = X$. This example induces the following definition:

Definition 2.1. *A topological space X will be caled a $CLCC$ – space if each countably compact subspace of X has countably compact closure.*

Remark. A : Let X be a countably infinite space with the discrete topology. It is clear that X is a $CLCC$ space but it is not pseudocompact.

B : The deleted Tychonoff plank $X = [0,\omega_1] \times [0,\omega_0] \setminus \{(\omega_1,\omega_0)\}$ is a pseudocompact space which is not countably compact. Furthermore, since $Y = [0,\omega_1) \times [0,\omega_0] \subset X$ is a countably compact (SCC, HCC) subset such that $cl_X(Y) = X$, it follows that X is not a $CLCC$ space.

The following propositions are straitforward.

Proposition 2.2. (a) *Every compact (HCC, SCC, ECC, ACC, sequentially compact, countably compact, normal, first countable, metrizable) space is $CLCC$.*

(b) Let Y be a countably compact subspace of $CLCC$ - space X and let $cl_X(Y) = X$. Then $CLCC$ property and countable compactness coincide in the space X.

Proposition 2.3. *Let X be a $CLCC$ space and Y be the closed subspace of X. Then Y is a $CLCC$ space.*

Proposition 2.4. *The discrete sum $\bigoplus\{X_s : s \in S\}$, where $X_s \neq \emptyset$ for every $s \in S$, has the $CLCC$ property if and only if all spaces X_s have the $CLCC$ property and the set S is finite.*

Proposition 2.5. *If X is a $CLCC$ space and Y is a compact space, then $X \times Y$ is a $CLCC$ space.*

Proof. Let Z be a countably compact subset of $X \times Y$ and $p_X(Z)$, $p_Y(Z)$ the projections of Z onto X, Y. Since the projections $p_X : X \times Y \longrightarrow X$, $p_Y : X \times Y \longrightarrow Y$ from $X \times Y$ onto X, Y are continuous mappings, the sets $p_X(Z)$, $p_Y(Z)$ are countably compact subsets of X, Y. Since X has the $CLCC$ property and Y has the compact property, the closure $cl_X(p_X(Z))$ is a countably compact subset of X and the closure $cl_Y(p_Y(Z))$ is a compact subset of Y. It is known that the Cartesian product $cl_X(p_X(Z)) \times cl_Y(p_Y(Z))$ of countably compact set $cl_X(p_X(Z))$ and compact set $cl_Y(p_Y(Z))$ is a countably compact subset of $X \times Y$. Furtrermore, the set $cl_X(p_X(Z)) \times cl_Y(p_Y(Z))$ is a closed subset of $X \times Y$ and $Z \subset cl_X(p_X(Z)) \times cl_Y(p_Y(Z))$. Since the countably compact property is hereditary with respect to closed sets, it is easy to see that the closure $cl_{X \times Y}(Z) \subset cl_X(p_X(Z)) \times cl_Y(p_Y(Z))$ is a countably compact subspace of $X \times Y$. ■

Remark. Let X be the product of spaces X_a, $a \in A$.

A: If every X_a, $a \in A$, is ECC space and A is a finite set, then X is a $CLCC$ space.

B: If every X_a, $a \in A$, is sequentially compact space and A is a countable set, then X is $CLCC$ space.

C: If every X_a, $a \in A$, is $HCC(SCC)$ space, then X is a $CLCC$ space.

Proposition 2.6. *Let $f : X \longrightarrow Y$ be a perfect mapping of a $CLCC$ space X onto a space Y. Then Y is a $CLCC$ space.*

Proof. Let P be a countably compact subset of Y. Since countably compactness is an inverse invariant of perfect mappings, the set $f^{-1}(P)$ is a countably compact subset of X. By the $CLCC$ property for the space X, the closure $cl_X(f^{-1}(P))$ is a countably compact subset of X. By perfectness, $f(cl_X(f^{-1}(P)))$ is a countably compact and closed subset of Y. Furthermore, the closure $cl_Y(P) \subset f(cl_X(f^{-1}(P)))$. Since countably compact property is hereditary with respect to closed sets, it follows that the closure $cl_Y(P) = cl_{f(cl_X(f^{-1}(P)))}(P)$ is a countably compact subset of Y. Hence the space Y has the $CLCC$ property.■

The following example shows that the continuous image of a $CLCC$ space need not be a $CLCC$ space.

Example 2.7. Let $X = [0,1]\times[0,\omega_1]\setminus\{(\alpha,\omega_1)\} : \frac{1}{3} < \alpha < \frac{2}{3}\}$ have the discrete topology and let $Y = [0,1] \times [0,\omega_1] \setminus \{(\alpha,\omega_1)\} : \frac{1}{3} \leq \alpha \leq \frac{2}{3}\}$ be the subspace of the product $[0,1] \times [0,\omega_1]$, where $[0,1]$ is with the usual topology and $[0,\omega_1]$ is the space of ordinals less then or equal to the first uncountable ordinal with the order topology. It is clear that the mapping $f : X \longrightarrow Y$ defined by $f(x) = x$, for all $x \in X$, is an continuous surjection. The space X has the $CLCC$ property, but Y is not a $CLCC$ space.

Remark. It is well known and easy to see that the following hold:

A: It is known that every first countable space is a continuous and open surjection of a metric space. Hence, by Proposition 2.2, every continuous and open image of a metric space is a $CLCC$ space.

B: Every normal space which has a unique compactification has the $CLCC$ property.

This is clear since every normal space which has a unique compactification is pseudocompact (and locally compact). It is known that in the class of normal spaces pseudocompactness and countable compactness coincide.

The following definition gives an information when spaces satisfy the $CLCC$- property.

Definition 2.8. *Let X be a topological space and $CC(X)$ the collection of all countably compact subsets of X. Then:*

(a) The space X is a $CLSC-$space if each member of $CC(X)$ has strongly countably compact (SCC) closure.

(b) The space X is a $CLHC-$ space if each member of $CC(X)$ has hyper-countably compact (HCC) closure.

(c) The space X is a $CLC-$ space if each member of $CC(X)$ has compact closure.

Remark. A : It is clear that $(c) \Rightarrow (b) \Rightarrow (a) \Rightarrow CLCC$ property.

B: Note that in the class of: compact spaces, metrizable spaces, Lindelöf spaces, σ-compact spaces, paracompact spaces, first countable spaces, realcompact spaces these properties coincide.

C: Let Y be a countably compact subspaces of a $CLSC$ ($CLHC$, CLC) space X and let $cl_X(Y) = X$. Then SCC property (HCC property, compactness) and $CLSC$ ($CLHC$, CLC) property coincide in the space X.

D: S.P.Franklin and M.Rajagopalan have constructed in [7] a compactification of integers with remainder homeomorphic to $[0, \omega_1]$. Removing $\{\omega_1\}$ gives the space $\gamma N \setminus \{\omega_1\}$ which is normal, separable, locally compact, first countable, sequentially compact but not strongly countably compact.

E: Propositions 2.3, 2.4 and 2.6 are true for (a), (b) and (c) property.

Proposition 2.9. *Let X be the product of space $X_a, a \in A$. If each $X_a, a \in A$, is a $CLSC$($CLHC$, CLC) space, then X is a $CLSC$ ($CLHC$, CLC) space.*

Proof. $CLSC$ case: Let Z be a countably compact subset of X and the sets $p_a(Z), a \in A$, projections of Z onto X_a. Since the projections $p_a : X \longrightarrow X_a$ from X onto $X_a, a \in A$, are continuous and open mappings, the sets $p_a(Z), a \in A$, are countably compact subsets of X_a. Every $X_a, a \in A$, have $CLSC$ property. Therefore, every subset $cl_{X_a}(p_a(Z)) \subset X_a, a \in A$, is SCC. Since SCC property is multiplicative, it follows that the product $\times\{cl_{X_a}(p_a(Z)) : a \in A\}$ is a SCC subspace of X. Furtrermore, the product $\times\{cl_{X_a}(p_a(Z)) : a \in A\}$ is a closed subset of X and $Z \subset \times\{cl_{X_a}(p_a(Z)) : a \in A\}$. Since SCC property is hereditary with respect to closed sets, it follows that the closure $cl_X(Z)$ is a SCC subset of X.

$CLHC$(CLC) case is proved in a similar way. ■

Corollary 2.10. *The inverse limit of an inverse system of $CLSC$ ($CLHC$, CLC) spaces is a $CLSC$ ($CLHC$, CLC) space.*

We close this section by the following question.

Question 2.11. Is it true that every normal and $CLSC$ space is $CLHC$?

3. Some properties similar to the Lindelöf property

Definition 3.1. *A space X is Menger(Rothberger) if for each sequence $(U_n, n \in N)$ of open covers of X, there exists a sequence $(V_n, n \in N)(V_n, n \in N)$, where for every $n \in N, V_n$ is a finite subfamily of $U_n(V_n \in U_n)$ and $\cup\{V_n : n \in N\} = X(\cup\{V_n : n \in N\} = X)$(see [9]).*

It is clear that every Rothberger space is a Menger space and every Menger space is a Lindelöf space.

In [11] and [13] we consider $L - spaces$ ($R - spaces$) [$L' - spaces$]. A topological space X is called an $L - space$ ($R - space$) [$L' - spaces$] if each countable [Lindelöf] subset of X has Lindelöf(realcompact) [Lindelöf] closure. In the class of $L - spaces$ ($R - spaces$) [$L' - spaces$] strongly countable compactness [hypercountable compactness] and countable compactness coincide. This criterion of countably compactness induce the following property.

Definition 3.2. *Let X be a topological space and $CC(X)$ the collection of all countably compact subsets of X. Then:*

(a) The space X is a $LCLCC$ – space if the closure of each member of $CC(X)$ has the Lindelöf property.

(b) The space X is a $MCLCC$ – space if the closure of each member of $CC(X)$ has the Menger property.

(c) The space X is a $RCLCC$ – space if the closure of each member of $CC(X)$ has the Rothberger property.

Remark. A : Observe a simple fact that any property from Definition 3.2 is an invariant of perfect mappings and is inherited by closed subspaces(the disjoint topological sum). Furthermore, the Cartesian product $X \times Y$ of a $LCLCC$($MCLCC$, $RCLCC$) space X and a compact space Y is $LCLCC$($MCLCC$, $RCLCC$).

B : By Definition 2.8, the space X is a $CLC - space$ if each member of $CC(X)$ has compact closure. It is shown that $CLC - property \Rightarrow (c) \Rightarrow (b) \Rightarrow (a)$.

Proposition 3.3. *In the class of Tychonoff spaces the properties (a), (b), (c) and CLC property coincide.*

Peoof. Case (a) $\Rightarrow$ CLC property: This is clear, because the closure of each member of $CC(X)$ is a Lindelöf (realcompact) and pseudocompact subspace. Therefore, each member of $CC(X)$ has compact closure. Hence $(a) \Rightarrow CLC$ property.■

Corollary 3.4. *In the class of first countable spaces properties (a), (b), (c), $CLC - property$ and isocompactness coincide*.*

*A space X is called an *isocompact space* ($ISCC - space$) if every closed countably compact subset of X is compact(see [11]).

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Viša hemijska-tehnološka škola
Narodnih heroja 10
15000 Šabac
Yugoslavia

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